Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII. AUGUST-SEPTEMBER, 1906. NOS. 8-9.

SOLUTION OF A PROBLEM IN THE THEORY OF NUMBERS.

By E. B. ESCOTT, Chicago, Ill.

PROBLEM.* Let p^n-1 and n have a common divisor d. In what case is $p^{n/\delta}-1$ divisible by $d(p^{n/\delta}-1)$, δ being a divisor of d? This is always true for the particular value $\delta=1$.

Let $p^{n/d}=P$ and $d=\delta a$. Then the problem may be stated as follows: Given that $P^{\delta a}-1$ is divisible by δa, to find the necessary and sufficient conditions that the expression $\dfrac{P^a-1}{\delta a(P-1)}$ shall be integral. [When the latter is integral, it is evident that the condition that $P^{\delta a}-1$ shall be divisible by δa is *a fortiori* satisfied].

Let $a=a_1{}^a a_2{}^b\ldots$, where a_1, a_2, ... are distinct primes. First, suppose that a_1 is a factor of $P-1$. Then $P=1+a^m\kappa$, where $m \gtreqless 1$ and κ is any integer. Raising both members of this equation to the power a_1, we get

$$P^{a_1}=1+a_1\ a_1{}^m\kappa+\ldots\equiv 1(\text{mod } a_1{}^{m+1}),$$

from which we see that $P^{a_1}-1$ is divisible by $a_1{}^{m+1}$ and by no higher power of a_1. Similarly, $P^{a_1{}^a}-1\equiv 0$ (mod $a_1{}^{m+a}$), and therefore P^a-1 is divisible by $a_1{}^{m+a}$ and by no higher power of a_1.

*Question 2932, *L' Intermédiaire des Mathématiciens*, 13 (1906), p. 87, proposed by L. E. Dickson. For application of this question, see an article by L. E. Dickson, *On finite algebras*, *Goettinger Nachrichten*, July, 1905.

Secondly, suppose that a_2 is not a factor of $P-1$. Let $P^e-1\equiv 0 \pmod{a_2}$, where e is the smallest exponent of P for which this congruence holds, and let P^e-1 be divisible by ${a_2}^l$, but by no higher power of a_2. Then, since it is necessary that $P^a-1\equiv 0 \pmod{{a_2}^b}$, if $l<b$, $P^{ea_2{}^{b-l}}-1\equiv 0 \pmod{{a_2}^b}$, while this is the lowest number of this form which is divisible by ${a_2}^b$. Therefore, it is necessary that a should be a multiple of $ea_2{}^{b-l}$, and since e must be a factor of a_2-1 by Fermat's Theorem, and is therefore relatively prime to a_2, e must be a divisor of $a/{a_2}^b$. It is, therefore, necessary and sufficient that a be divisible by e.

If $l>b$, P^e-1 is the lowest number of this form which is divisible by ${a_2}^b$, and we have as before that the necessary and sufficient condition is that a shall be divisible by e.

It can be shown that δ and $P-1$ can have no common factor. If they had a common prime factor δ_1, then since

$$\frac{P^a-1}{P-1}=P^{a-1}+P^{a-2}+\ldots+P+1\equiv +1+\ldots+1\equiv a \pmod{\delta_1},$$

δ_1 must be a factor of a. Suppose that $\delta_1=a_1$. Then, since the numerator P^a-1 is divisible by ${a_1}^{m+a}$ and by no higher power, while in the denominator a contains ${a_1}^a$ and $P-1$ contains ${a_1}^m$, and δ contains a, to at least the first power, then the denominator would contain ${a_1}^{m+a+1}$, so the fraction could not be equal to an integer.

Theorem. *The necessary and sufficient conditions that* $(P^a-1)/\delta a(P-1)$ *shall be integral are:* (*1*) a *must be divisible by* e, *the least integer such that* P^e-1 *is divisible by* a_κ, *where for* a_κ *is taken in turn the various prime factors of* a *not dividing* $P-1$; (*2*) δ *is any divisor of* $(P^a-1)/a(P-1)$.

Example. Let $P=3^6$. Then $P-1=2^3.7.13$. Let $a_1=7$, $a_2=23$. Then $e=11$, since $P^{11}\equiv 1 \pmod{23}$. Therefore a must be a multiple of 7, 23, and 11. For example, let $a=7.11.23$. Then δ may be taken as any divisor of $\frac{3^{6.7.11.23}-1}{7.11.23(3^6-1)}$; for example, $\delta=23$, 67, 547, 661, 1093, 3851, ..., or various numbers composed of these factors.

NOTE ON CERTAIN QUADRATIC NUMBER SYSTEMS FOR WHICH FACTORIZATION IS UNIQUE.

By G. B. BIRKHOFF.

If we define w either as a root of an equation

$$w^2=\frac{D}{4}=0\ldots(1),$$